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Limits to the Acceleration of Black Holes

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Using the Dirac equation but with a null surface condition, we obtain the event horizon equation, and the temperature function of a rectilinearly accelerating black hole with electric-magnetic charge. The cosmic censorship hypothesis implies that the acceleration of a black hole has a limit, which is M/r_H^2 for the rectilinearly accelerating black hole. The limit values of acceleration for other black holes, including the rectilinearly accelerating black hole with electric-magnetic charge the Kerr–Newman black hole, and the Kerr black hole are given.

1. INTRODUCTION

In 1905, Einstein established the special theory of relativity based on the principle of the constancy of light velocity, according to which the limiting speed is the speed of light. Does acceleration have a limit? We have investigated several accelerating black holes (BH) and find that the acceleration of each type has a limit, which is M/r_H^2 for the rectilinearly accelerating black hole, for example. We will study several common accelerating BHs and give the limit value for them. First we deal with the rectilinearly accelerating BH with electric-magnetic charge.

2. THE LIMIT VALUE FOR RECTILINEARLY A ACCELERATING BLACK HOLE WITH ELECTRIC–MAGNETIC CHARGE

In the metric given in ref. 1, if $b = c = \lambda = 0$, the line element in the space-time of a rectilinearly accelerating BH with electric–magnetric charge is given by

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$$ds^2 = (H - r^2a^2 \sin^2\theta)dv^2 - 2dv\, dr + 2r^2 a \sin\theta\, dv\, d\theta - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{1}$$

where

$$H = 1 - 2M/r - 2ar\cos\theta + (E^2 + Q^2)/r^2 - [4a(E^2 + Q^2)\cos\theta]/r \tag{2}$$

and $M = M(v)$, $E = E(v)$, and $Q = Q(v)$ are, respectively the mass, electric charge, and magnetic charge of the BH, $a = a(v)$ is the value of the acceleration, and V is the Eddington advanced time coordinate.

In this space-time, the 4-component spinor equation of Dirac particles can be written as [2]

$$(D + \varepsilon - \rho + i\alpha_0 U_\mu L^\mu)F_1 + (\bar{\delta} + \pi - \alpha + i\alpha_0 u_\mu \bar{m}^\mu)F_2 - \frac{i}{\sqrt{2}}\mu_0 G_1 = 0$$

$$(\Delta + \mu - \gamma + i\alpha_0 U_\mu n^\mu)F_2 + (\delta + \beta - \tau + i\alpha_0 U_\mu m^\mu)F_1 - \frac{i}{\sqrt{2}}\mu_0 G_2 = 0$$

$$(D + \varepsilon^* - \rho^* + i\alpha_0 U_\mu L^\mu)G_2 - (\delta + \pi^* - \alpha^* + i\alpha_0 U_\mu m^\mu)G_1 - \frac{i}{\sqrt{2}}\mu_0 F_2 = 0$$

$$(\Delta + \mu - \gamma^* + i\alpha_0 u_\mu n^\mu)G_1 - (\bar{\delta} + \beta^* - \tau^* + i\alpha_0 u_\mu \bar{\mathrm{m}}^\mu)G_2 - \frac{i}{\sqrt{2}}\mu_0 F_1 = 0 \tag{3}$$

where F_1, F_2, G_1, and G_2 are 4-component spinosrs, α_0 and μ_0 are, respectively the electric–magnetic charge and the rest mass of the particle. $U_\mu = (U_v, U_r, u_\theta, U_\varphi)$ is the 4D electromagnetic potential. It can be proved that only $U_\varphi = 0$. Here L_u, n_u, m_u, and $\bar{m}_u$ is a null tetrad set given as follows, which satisfies the pseudoorthogonal condition and the metric condition [3]:

$$L^\mu = (0, -1, 0, 0)$$
$$n^\mu = (1, \tfrac{1}{2}(H - r^2a^2\sin^2\theta), 0, 0)$$
$$m^\mu = (0, -ra\sin\theta/\sqrt{2}, (\sqrt{2}r)^{-1}, i(\sqrt{2}r\sin\theta)^{-1}) \tag{4}$$

D, $\mathfrak{D}$, δ and $\bar{\delta}$ are ordinary differentiation operators; ε, pgr, etc., are the spin coefficients. According to a method due to Newman and penrose [3], we obtain the differentiation operators and the spin coefficients as follows:

$$D = -\partial/\partial r, \qquad \Delta = \partial/\partial V + \tfrac{1}{2}(H - r^2a^2\sin^2\theta)\partial/\partial r$$
$$\delta = -ra\sin\theta/\sqrt{2}\cdot\partial/\partial r + 1/(\sqrt{2}r)\cdot\partial/\partial\theta + i/(\sqrt{2}r\sin\theta)\cdot\partial/\partial\varphi$$
$$\varepsilon = 0, \qquad \rho = -1/r, \qquad \tau = -\pi = a\sin\theta/(2\sqrt{2}r)$$
$$\alpha = (a\sin\theta - \mathrm{ctg}\,\theta/(2r))/\sqrt{2}, \qquad \beta = \mathrm{ctg}\,\theta/(2\sqrt{2}r)$$
$$\mu = (H - r^2a^2\sin^2\theta)/(2r), \qquad \gamma = -\tfrac{1}{4}\partial H/\partial r + r^2a^2\sin^2\theta/2 \tag{5}$$

Separating variables

$$F_j = e^{im\varphi} f_j(\nu, r, \theta), \qquad G_j = e^{im\varphi} g_j(\nu, r, \theta), \quad j = 1, 2 \tag{6}$$

where m is the magnetic quantum number, then substituting (4)–(6) and $U = (U_v, U_r, U_\theta, U_\varphi)$ into Eq. (13), we obtain

$$\begin{aligned}
&\sqrt{2} r D_1 f_1 - (L_+ - ra \sin\ \theta) f_2 + i\mu_0 r g_1 = 0 \\
&\sqrt{2} r D_2 f_2 + L - f_1 - i\mu_0 r g_2 = 0 \\
&\sqrt{2} r D_1 g_2 + (L_- - ra \sin\theta) g_1 + i\mu_0 r f_2 = 0 \\
&\sqrt{2} r D_2 g_1 - L_+ g_2 - i\mu_0 r f_1 = 0
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
D_1 &= \partial/\partial r + 1/r \\
D_2 &= \partial/\partial v + \tfrac{1}{2}(H - r^2a^2 \sin^2\theta) \cdot \partial/\partial r + (H - r^2a^2 \sin^2\theta/(2r) \\
&\quad + 1/4 \cdot \partial R/\partial r - ra^2 \sin^2\theta/2 \\
L_\pm &= -r^2 a \sin\theta \cdot \partial/\partial r + \partial/\partial\theta \pm m/\sin\theta + \mathrm{ctg}\ \theta/2 \ \ - ra \sin\theta
\end{aligned} \tag{8}$$

In order to investigate the quantum behavior of Dirac particles near the event horizon, we introduce the following generalized tortoise coordinate transformation [4]:

$$\begin{aligned}
r_* &= r + 1/2\ \ k(v_0, \theta_0) \cdot \ln\left[r - r_H(v, \theta)\right] \\
v_* &= v - v_0, \qquad \theta_* = \theta - \theta_0
\end{aligned} \tag{9}$$

Then, D_1, D_2 and $L_\pm$ can be written as

$$\begin{aligned}
D_1 &= \frac{\rho + 1}{\rho} \cdot \frac{\partial}{\partial r_*} + \frac{1}{r} + i\alpha_0 U_r \\
L_\pm &= -\frac{\Delta}{\rho} \cdot \frac{\partial}{\partial r_*} + \frac{\partial}{\partial\theta_*} + \Sigma_\pm \\
D_2 &= \frac{\partial}{\partial v_*} + \frac{x}{\rho} \cdot \frac{\partial}{\partial r_*} + Y
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&X = \tfrac{1}{2}(H - r^2a^2 \sin^2\theta)(\rho + 1) - \dot{r}_H \\
&\Delta = r^2 a \sin\theta\ (\rho + 1) + r_{H,\theta} \\
&\rho = 2k(r - r_H), \qquad \dot{r}_H = \partial r_H/\partial v, \qquad r_{H,\theta} = \partial r_H/\partial\theta
\end{aligned} \tag{11}$$

While (8) and (10) are defined as generalized momentum operators, (10) gives the generalized momentum operators near the event horizon, which determines the quantum behavior of a Dirac particle enar the event horizon.

Now we study the coefficients of $\partial/\partial r_*$ in formula (10). It is reasonable to require that X/ρ and Δ/ρ keep infinite when $r \to r_H$. Then it follows that

$$\lim_{r \to r_H} \frac{\Delta}{\rho} = \text{limited value} \tag{12}$$

$$\lim_{r \to r_H} \frac{x}{\rho} = A_0 \text{ (finite)} \tag{13}$$

From (12) we obtain

$$r_H^2\, a \sin\theta + r_{H,\theta} = 0 \tag{14}$$

and from (13), we obtain

$$(H - r^2 a^2 \sin^2\theta - 2\dot{r}_H)\big|_{r \to r_H} = 0 \tag{15}$$

From (14) and (15), we obtain the event horizon equation,

$$1 - \frac{2M}{r_H} - 2ar_H \cos\theta + (E^2 + Q^2)/r_H^2 - 4a(E^2 + Q^2)\cos\theta/r_H - 2\dot{r}_H$$

$$+ 2a \sin\theta\, r_{H,\theta} + \frac{r_{H,\theta}^2}{r_H^2} = 0$$

By L'Hôspital rule, calculating the limit of (13), one can get the temperature function of a black hole near the event horizon as follows:

$$k = \frac{M/r_H^2 - a\cos\theta - (E^2 + Q^2)/r_H^3 + 2a(E^2 + Q^2)\cos\theta/r_H^2 - r_H a^2 \sin^2\theta}{A_0 - (H - r_H^2 a^2 \sin^2\theta)} \tag{16}$$

When $a = 0$, the value of k should be the same as for an evaporating black hole with electric–magnetic charge:

$$k = [M - (E^2 + Q^2)/r_H]/[2Mr_H - (E^2 + Q^2)] \tag{17}$$

Comparing formula (16) with (17,), we obtain $A_0 = 1$. So,

$$k = \frac{M/r_H^2 - a\cos\theta - (E^2 + Q^2)/r_H^3 + 2a(E^2 + Q^2)\cos\theta/r_H^2 - r_H a^2 \sin^2\theta}{2M/r_H + 2ar_H\cos\theta - (E^2 + Q^2)/r_H^2 + 4a(E^2 + Q^2)\cos\theta/r_H + r_H^2 a^2 \sin^2\theta} \tag{18}$$

By use of (14) it can be proved that this result is the same as formula (19) in ref 5. To calculate the partial derivative of k with respect to r_H, and letting $\partial k/\partial r_H = 0$, we have

$$k = \frac{2a \cos\theta\, r_H - 3a^2 \sin\theta\, r_H^2 + (E^2 + Q^2)/r_H^2}{2M + 6a \cos\theta\, r_H^2 + 4a(E^2 + Q^2) \cos\theta + 4a^2 \sin^2\theta\, r_H^3} \tag{19}$$

we see that the bigger r_H is the smaller k is. r_H has a maximum value when $\theta = 0$, while k is of minimum value [5]. From (18), we have

$$k\Big|_{\theta=0} = \frac{M/r_H^2 - a - (E^2 + Q^2)/r_H^3 + 2a(E^2 + Q^2)/r_H^2}{2M/r_H + 2ar_H - (E^2 + Q^2)/r_H^2 + 4a(E^2 + Q^2)r_H} \tag{20}$$

Similarly, when $\theta = \pi$, r_H has a minimum value, while k is of maximum value; then

$$k\Big|_{\theta=\pi} = \frac{M/r_H^2 + a - (E^2 + Q^2)/r_H^2 - 2a(E^2 + Q^2)/r_H^2}{2M/r_H - 2ar_H - (E^2 + Q^2)/r_H^2 - 4a(E^2 + \mathrm{R}Q^2)/r_H} \tag{21}$$

In (20), if $a = [M/r_H^2 - (E^2 + Q^2)/r_H^3]/[1 - 2(E^2 + Q^2)/r_H^2]$, then $k_\theta = 0$. This would violate the cosmic censorship hypothesis. So it is impossible that the value of acceleration for a rectilinearly accelerating BH with electric–magnetic charge should reach

$$a_{(0)} = [M/r_{H(0)}^2 - (E^2 + Q^2)/r_{H(0)}^3]/[1 - 2\,(E^2 + Q^2)/r_{H(0)}^2] \tag{22}$$

In (21), if

$$a_{(\pi)} = [M/r_{H(\pi)}^2 - (E^2 + Q^2)/2\; r_{H(\pi)}^3]/[1 + 2(E^2 + Q^2)/r_{H(\pi)}^2/] \tag{23}$$

then $k\big|_{\theta=\pi} = +\infty$, which is also impossible. That would make the mass of the BH gush from its tail and then proceed to blow up. In (22) and (23), $r_{H(0)}$ and $r_{H(\pi)}$ denote, respectively, the radius at the points $\theta = 0$ and $\theta = \pi$ of the BH. Because of

$$E^2 + Q^2 << M, \qquad r_{H(0)} > r_{H(\pi)}$$

we have $a_{(0)} < a_{(\pi)}$. Therefore the acceleration for a rectilinearly accelerating BH with electric–magnetic charge cannot reach $a_{(0)}$, which is expressed in (22).

Note that $a_{(0)}$ and $a_{(\pi)}$ are not relative to the angle θ. Because of straight-line motion, and no rotation, the values of acceleration of the BH are the same at different points on the event horizon. On the right-hand side of (22), there exists a factor $r_{h(0)}$, but this does not show that $a_{(0)}$ is only the acceleration of the BH at the point $\theta = 0$. (22), only shows that if the acceleration of the BH takes the value shown on the right-hand side, there would be $k = 0$.

3. THE RECTILINEARLY ACCELERATING BLACK HOLE

In the line element (1), if $E = Q = 0$, we have

$$d^2 = (1 - \frac{2M}{r} - 2ar\cos\theta - r^2a^2\sin^2\theta)\,dv^2 - 2\,dv\,dr - 2r^2a\sin\theta\,dv\,d\theta$$
$$- r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{24}$$

According to the above method, we get

$$k = [m/r_H^2 - a\cos\theta - a^2r_H\,\mathrm{Sin}^2\theta]/[2M/r_H + 2ar_H\cos\theta + r_H^2a^2\sin^2\theta] \tag{25}$$

and

$$1 - 2M/r_H - 2ar_H\cos\theta - 2\dot{r}_H - r_H^2a^2\sin^2\theta = 0 \tag{26}$$

By use of

$$r_{H,\theta} + r_H^2a\sin\theta = 0$$

we transform (26) into

$$1 - \frac{2M}{r_H} - 2ar_H\cos\theta - 2\dot{r}_H + 2ar_{H,\theta}\sin\theta + r_{H,\theta}^2/r_H^2 = 0 \tag{27}$$

and (25) into

$$k = \frac{M/r_H^2 - a\cos\theta - (r_{H,\theta})^2/r_H^3}{4M/r_H + 4a\cos\theta r_H - 2a\sin\theta(r_{H,\theta}) + 2\dot{r}_H - 1} \tag{28}$$

Equations (27) and (28) are respectively the event horizon equation and the temperature function near the event horizon of a rectilinearly accelerating BH. From (28), we see that the bigger r_H is, the smaller the value of the temperature is. At point $\theta = 0$, the value of r_H is biggest, so the value of k is smallest, given by

$$k\Big|_{\theta=0} = [M/r_{H(0)}^2 - a]/[2M/r_{H(0)} + 2ar_{H(0)}] \tag{29}$$

At the point $\theta = \pi$, the value of r_H is smallest, so the value of k is biggest given by

$$k\Big|_{\theta=\pi} = [M/r_{H(\pi)}^2 + a]/[2M/r_{H(\pi)} - 2ar_{H(\pi)} \tag{30}$$

where $r_{H(0)}$ and $r_{H(\pi)}$ are respectively the values of the radius of the event horizon at the points, $\theta = 0$ and $\theta = \pi$. If $a = 0$, then $r_{H(0)} = r_{H(\pi)}$, while

$$K\Big|_{\theta=0} = k\Big|_{\theta=\pi} = \frac{1}{2r_H} = \frac{1 - 2\dot{r}_H}{4M} \tag{31}$$

This just is the temperature near the event horizon of a Vaidya BH.

If $a = M/r_{H(0)}^2$, and $a = M/r_{H(\pi)}^2$, then $k|_{\theta=0} = 0$ and $k|_{\theta=\pi} = +\infty$. This is impossible. Because of $r_{H(0)} > r_{H(\pi)}$, $M/r_{H(0)}^2 < M/r_{H(\pi)}^2$, so we draw the following conclusion: It is impossible for the value of acceleration for a rectilinearly accelerating BH to reach $M/r_{H(0)}^2$.

This shows that the value of the acceleration for a rectilinearly accelerating BH has a limit, which is $M/r_{H(0)}^2$.

4. KERR–NEWMAN BLACK HOLE

The line element in the space-time of the Kerr–Newman BH is

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[\left(r^2 + a^2\right) \sin^2\theta + \frac{(2mr - Q^2)a^2 \sin^4\theta}{\rho^2}\right] d\varphi^2 - \frac{2(2Mr - Q^2)\, a \sin^2\theta}{\rho^2} dt\, d\varphi \quad (32)$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta \quad (33)$$

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \quad (34)$$

M and q are, respectively, the mass and the charge of the BH, and $a = J/M$ is the angular momentum of a unit mass. There are two event horizons, which are given by

$$r_H^{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2} \quad (35)$$

The temperature function is

$$k = \frac{r_H^+ - r_H^-}{2(r_+^2 + a^2)} \quad (36)$$

which can be reduced to

$$k = \left[2M + \frac{2M^2 - Q^2}{\sqrt{M^2 - a^2 - Q^2}}\right]^{-1} \quad (37)$$

We see that if

$$a = \sqrt{M^2 - Q^2} \quad (38)$$

then $k = 0$, which violates the third law of thermodynamics. If

$$a > \sqrt{M^2 - Q^2}$$

then $r_H^{\pm} = M \pm i\sqrt{a^2 - (M^2 - Q^2}$, the event horizon of BH would disappear, and the odd-circle would appear, which violates the cosmic censorship hypothesis. So $a < \sqrt{M^2 - Q^2}$. We have

$$a = J/M = \omega r_H^2 \tag{39}$$

where ω is the angular speed. The centripetal acceleration is

$$a_n^2 = \omega^2 a \tag{40}$$

One sees that a is not the rotational acceleration, but it determines the value of centripetal acceleration when ω is finite. Therefore, it is reasonable to regard a as the acceleration. From (39), (40), we have

$$a_n = a^2/r_H^3 < (M^2 - Q^2)/r_H^3 \tag{41}$$

which is the limit value of acceleration of the Kerr–Newman BH.

5. KERR BLACK HOLE

The line element in the space-time of the Kerr BH is

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2\, d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2} \right] d\varphi^2 - \frac{4Mra\sin^2\theta}{\rho^2}\, dt\, d\varphi \tag{42}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{43}$$

$$\Delta = r^2 + a^2 - 4Mr \tag{44}$$

It has two event horizons:

$$r_H^{\pm} = M \pm \sqrt{M^2 - a^2} \tag{45}$$

The surface gravity is

$$k = \sqrt{M^2 - a^2}/2\,(M^2 + M\sqrt{M^2 - a^2}) \tag{46}$$

We see that when a is fixed, k has a fixed value. When $a = M$, $k = 0$, which is impossible for violating the third law of thermodynamics. For this reason, the value of a cannot come up to M, that is, the centripetal acceleration of a Kerr BH in self-rotation has a limt, which is

$$a_n = M^2/r_H^2 \tag{47}$$

where a_n is the centripetal acceleration on the event horizon.

6. CONCLUSION

We have studied four kinds of accelerating black holes, and find that their accelerations are limited. We draw the following conclusion: The limiting accelerations are, respectively, $[M/r_{H(o)}^2 - (E^2 + Q^2)/r_{H(0)}^2] / [1 - 2(E^2 + Q^2)/r_{H(0)}^2]$, $M/r_{H(0)}^2$, $(M^2 - Q^2)/r_H^3$, and M^2/r_H^3 for the rectilinearly accelerating BH with electric–magnetic charge, the rectilinearly accelerating BH, the Kerr–Newman BH, and the Kerr BH.

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